Selection Rule for Generation Numbers and Gauge Anomalies for Unification Groups

Huazhong Zhang

Theoretical Physics Group
Lawrence Berkeley Laboratory
MS 50A-3115, 1 Cyclotron Road
Berkeley, California 94720, USA.[1]

and
P. O. Box 17660
Jackson State University
Jackson, MS 39217, USA[2]
(e-mail:fnalv::cpzhang;hzhang@riscman.jsums.edu)

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Abstract

The possibilities of global (non-perturbative) gauge anomalies for a class of gauge groups are investigated. Intimately connected to branching rules and topological aspect of gauge groups, the results are applied to the study of unification gauge groups such as SO(10), SU(5), E_6, E_8 etc. Especially, we discuss extensively about the selection rule for generation numbers $N_f + N_{mf} = even \geq 4$ in SO(10) and supersymmetric SO(10) unification theories as originally proposed by the author[1], where N_f and N_{mf} denote the generation numbers for ordinary fermions and mirror fermions respectively. This is due to the global gauge anomalies from some subgroups of SO(10) in a fundamental spinor representation such that the ill-defined 'large' gauge transformations in the subgroup cannot be unwrapped in SO(10) in the quantum theory as we noted[1]. A similar result related to left-right symmetric models is also given. PACS: 02.20.+b, 02.40.+m, 11.15.-q.

1 Introduction

Symmetries and the properties of Lie groups have been always intriguing in physics, especially in the understanding of fundamental interactions. In this connection, as a matter of fact, non-abelian gauge theories have been the main frame-work in elementary particle physics since the formulation of Yang-Mills theories[2]. Lie algebras and topological properties of Lie groups have played important

[1]Where the main work supported by DOE
[2]Permanent address

roles in non-abelian gauge theories. In particular, the topological properties of Lie groups have been shedding new lights on the study of non-abelian gauge theories in the topological and non-perturbative aspects. In this paper, we will study some non-perturbative aspects for unification gauge groups.

Since the building of standard electroweak gauge theory[3], one of the most interesting ideas in particle physics has been incorporating the standard model into a grand unified theory[4−5] (GUT) or a supersymmetric grand unified theory[6−7]. Although the minimal SU(5) model[4] does not lead to the desired unification or is not compatible with proton decay search[8] and CERN LEP data[9], it is now known that the unification may be achieved in either a supersymmetric GUT, or a GUT with a gauge group larger than SU(5) spontaneously broken to the standard gauge group in at least two stages. Such an example[10] is the SO(10) GUT model which may break[11] first to left-right symmetric $SU(3) \otimes SU(2)_L \otimes SU(2)_R \otimes U_{B-L}$ model at a scale M_X and then to the standard model. Gauge hierarchy problem can be naturally solved in supersymmetric SO(10) theory, without evidence of supersymmetry at low energies, non-supersymmetric SO(10) models are still of interest[11]. The SO(10) models have many attractive physics features[12] such that they preserve a good prediction for the $\sin^2\theta_w$ for Winberg angle, permit neutrinos having small masses through see-saw mechanism, and may give predictions for some parameters in the standard model. One of the interesting features in SO(10) unification models and supersymmetric SO(10) unification models lies at the structure of the group representation itself. That is, the chiral states of quarks and leptons in a family including a right-handed neutrino can be fitted neatly into a fundamental spinor representation (f.s) of dimension 16 for the SO(10) gauge group. One of the main purposes of the present paper is to discuss extensively about this group theoretic structure and the relevant physics effects. Especially, we will focus on its global (non-perturbative) effects. Then, related remarks and consequences will be given. As a matter of fact, as we will see that the SO(10) models with Weyl fermions in a fundamental spinor representation will generate a new type of global (non-perturbative) gauge anomalies[1] when restricting to some gauge subgroups with non-trivial forth homotopy group. The ill-defined large gauge transformations in the subgroups cannot be unwrapped in SO(10). Therefore, the corresponding generating functional in the restricted subgroup sector is ill-defined, or the quantum theory is inconsistent. This also suggests that the possibilities for this type of global gauge anomalies need to be taken into consideration carefully in general in non-abelian gauge theories with Weyl fermions.

Our discussions will be organized as follows. In the next section, we will first give a brief description of global (non-perturbative) gauge anomalies in gauge theories. Then in section 3, we will clarify the global gauge anomalies for some gauge groups as the direct product of SU(2) or SP(2N). In section 4, the results will then be applied to the discussions of SO(10) as well as other unification groups and the selection rule for generation numbers as originally proposed[1] by the author. Our conclusion will be summarized in section 5.

2 Global (Non-perturbative) Gauge Anomalies in Even Dimensions

In this section, we will give a brief description of global gauge anomalies in even dimensions needed for our discussions in the paper. We refer the details to the relevant references.

Gauge anomalies arise as the gauge transformations which are classically well-defined become inconsistent in the quantum theory. For a gauge theory with Weyl fermions in 2n dimensions, if the homotopy group $\Pi_{2n}(G)$ is non-trivial for the relevant gauge group G, then there can be topologically non-trivial and continuous gauge transformations on the compactified spacetime manifold. If

such topologically non-trivial gauge transformations generate inconsistency in the quantum theory, then the anomalies are global (non-perturbative). The gauge anomalies corresponding to the topologically trivial gauge transformations are perturbative or local. In general, for a non-abelian gauge theory in an even dimensions D=2n, one still needs to consider the possibility of gauge anomalies if the homotopy group $\Pi_{2n}(G)$ for the gauge group G is non-trivial when the theory is free of local (perturbative) gauge anomalies.

It was shown by Witten[13] that an SU(2) gauge theory in four dimensions with an old number of Weyl fermion doublets is mathematically inconsistent due to a global (non-perturbative) gauge anomaly as the sign change of the fermion measure for the large gauge transformations. With a global gauge anomaly, the generating functional for the quantum theory is ill-defined. Topologically, this is associated with the fact that the homotopy group $\Pi_4(SU(2)) = Z_2$ is non-trivial. Global gauge anomalies have been investigated for SU(N) gauge groups[14−21], and systematically and rather generally for arbitrary compact and connected simple gauge groups in generic even dimensions[15−21], especially[15−21] in terms of the James numbers of Stiefel manifolds and generalized Dynkin indices. The study of global gauge anomalies are meaningful only if the gauge theory is free of local (perturbative) gauge anomalies, since otherwise the theory is anomalous even for infinitesimal gauge transformations. Furthermore, only if the infinitesimal gauge transformations are well-defined, topologically the homotopy group for gauge transformations can be well-defined since the homotopic equivalence is defined modulo infinitesimal gauge transformations.

For local gauge anomalies, there can be Green-Schwarz mechanism[22] of anomaly cancelation when gravitational field etc are also included in the theory. Witten[23] and others[24] derived a general formula for global gauge anomalies including gravitation by index theorem. But the quantities in the formula in general do not have a convenient scheme for explicit calculations. However, for pure gauge theories, one is usually interested in the strong anomaly cancellation condition $TrX^{n+1} = 0$ for a generic Lie algebraic element X for the group under consideration. The method developed in Refs.14-21 are more convenient and can be implemented explicitly in many and rather general cases to calculate the global gauge anomaly coefficient. In particular, the global gauge anomalies expressed in terms of[16] the James numbers of Stiefel manifold and generalized Dynkin indices are demonstrated powerful so that many classes of gauge theories in 2n dimensions have been studied extensively. The possible global gauge anomalies for many generic groups in rather generic dimensions have been determined completely in this approach[15−21]. The essential idea in this approach can be briefly described as follows. Consider a gauge theory of gauge group H in 2n dimensions with Weyl fermions in its representation ω free of local gauge anomaly. Let $\Pi_{2n}(H) \neq 0$, so that the theory may possibly possess a global gauge anomaly. Now consider further a gauge group G such that H is a subgroup of G and $\Pi_{2n}(G) = \{0\}$. Then the group G can be regarded as a principal bundle over G/H with H as the fiber and the structure group. The fibration $H \rightarrow G \rightarrow$G/H leads to the exact homotopy sequence[25]

$$... \rightarrow \Pi_{2n+1}(G) \rightarrow \Pi_{2n+1}(G/H) \rightarrow \Pi_{2n}(H) \rightarrow \Pi_{2n}(G) = 0. \quad (1)$$

Therefore, the global anomaly of H may be calculated as the local anomaly of G. It appears as the integration for the corresponding Wess-Zumino term over a (2n+1)-dimensional disc D^{2n+1} with the compactified space time S^{2n} as its boundary. In order for this method to work, one needs to find a rep. $\tilde{\omega}$ of G such that when G is restricted to H on the spacetime 2n sphere S^{2n} as the boundary of the D^{2n+1}, the $\tilde{\omega}$ reduces to the ω for H plus H singlets. As emphasized in ref.15-16, usually such a rep condition can be realized only in the generalized convention of allowing negative multiplicities for Weyl fermions of different chirality. For details of the analysis in the approach, see the refs. 14-21. For many groups in arbitrary 2n dimensions, explicit formulas and rather general results can be obtained in this approach. As an example, we have obtained the general formula for

the global gauge anomaly coefficient for SU(N) groups as expressed in the following proposition[16].

Proposition 1. The global anomaly coefficient $A(\omega)$ for a rep. ω of SU(n-k) $(0 \leq k \leq n-2)$ in D=2n dimensions is given by

$$A(\omega) = exp\{\frac{2\pi i}{d_{n+1,k+1}} Q_{n+1}(\tilde{\omega})\}, \tag{2}$$

where

$$d_{n+1,k+1} = \frac{n!}{U(n+1,k+1)} = integers. \tag{3}$$

The integral number U(n+1, k+1) is the James number for the complex Stiefel manifold, SU(n+1)/SU(n-k), and the $Q_{n+1}(\tilde{\omega})$ is the (n+1)-th Dynkin index for the $\tilde{\omega}$.

For details of the derivation and discussions of the formula and James numbers of Stiefel manifolds, see ref.16-20. For the sake of the later sections, in the following we will give some of our results[15−21] as propositions and brief clarification relevant to our discussions in the present paper.

Proposition 2. (i) Any irreducible representation (irrep) ω of SU(2) has no global anomaly in D=0(mod 8) dimensions. (ii) Only spinor reps of SU(2) with spins $J = \frac{1}{2}(4l+1) = \frac{5}{2}, \frac{9}{2}, ...$, have Z_2 global anomalies in D=4(mod 8). Neither reps with $J = \frac{1}{2}(4l+3) = \frac{3}{2}, \frac{7}{2}, ...$, nor reps with integer J have global anomalies.

Generally for SP(2N) (N=rank) groups, we have the following result.

Proposition 3. (i) Any rep ω of SP(2N) has no global as well as local gauge anomalies in dimensions D=0(mod 8) (ii) Any locally anomaly-free rep ω of SP(2N) has no global anomaly in D=2 or 6 (mod 8). (iii) The global anomaly coefficient A(ω) of SP(2N) in dimension D=4(mod 8) is given by

$$A(\omega) = exp[i\pi Q_2(\omega)], \tag{4}$$

where $Q_2(\omega)$ is the 2nd Dynkin index[26,27] normalized to $Q_2(\Box) = 1$ for the 2N-dimensional fundamental rep, and given by[27] in general

$$Q_2(\omega) = \frac{d(\omega)}{2N(2N+1)} \sum_{j=1,N} \{(l_j)^2 - (l_j^{(0)})^2\}, \tag{5}$$

with

$$l_j = f_j + N - j + 1, \ l_j^{(0)} = N - j + 1, (1 \leq j \leq N), \tag{6}$$

for the Young tableau $\Gamma = \{f_1, f_2, ..., f_N\}$ satisfying $f_1 \geq f_2 \geq ... \geq f_N \geq 0$ corresponding to ω.

In particular, for SU(2)=SP(2) with N=1, $f_1 = 2J$ (J=0, 1/2, 1, 3/2...) we have

$$Q_2(\omega) = \frac{2}{3} J(J+1)(2J+1), \tag{7}$$

One can easily see that the Proposition 2 is a special case of the Proposition 3, we list the SU(2) case separately due to its special interest and importance.

Proposition 4. In arbitrary D=2n dimensions, if the relevant Weyl fermion rep $\tilde{\omega}$ of G free of local gauge anomaly in the strong anomaly cancellation condition $TrX^{n+1} = 0$ reduces to an *irreducible* rep ω of H plus H singlets, then there will be no H global gauge anomalies for the rep ω.

Remark. Note that the Proposition 4 applies to semisimple gauge groups H and G also if both the topological and rep conditions are satisfied at least in the case that $\Pi_{2n+1}(G/H)$ does not contain more than one infinite cyclic Z's, since there may be only a unique Wess-Zumino term in this case. The idea of global gauge anomaly appearing as local in a larger group may determine effectively the possible global gauge anomalies for the H gauge group. In the case of more than one Wess-Zumino terms involved, for instance, if the ω and $\tilde{\omega}$ are not an irrep plus singlets, this method may not apply in general for non-simple groups. An exceptional case will be seen later in which there are more than one Wess-Zumino terms involved but the exact homotopy sequence may be regarded as the direct sum of more than one due to the fact that the simple ideals in H are effectively embedded into the corresponding simple ideals of G respectively, so that each of them for the direct sum of the exact homotopy sequence then may be regarded as corresponding to only one topologically independent Wess-Zumino term for a simple ideal pair in H and G of the embedding. For a simple gauge group[16], the above proposition applies to a generic rep ω as we have noted. In the case of $\omega = \sum_i \oplus \omega_i$ for a simple H, the Wess-Zumino term is topologically unique for the embedding of H into G, although formally it may have more than one terms from different irreps in the direct sum.

Before going to the next section, we also note the following facts. It is well known that[28,29] local gauge anomalies can only arise from Weyl fermions in the complex reps of SU(N) ($N \geq 3$). Topologically, this is due to the fact that $\Pi_5(G)$ can be infinite cyclic only for G=SU(N) ($N \geq 3$). The groups SO(4k+2) (k=integer) and E_6 cannot have perturbative gauge anomalies although they also have complex reps, since the $\Pi_5(G)$ is trivial for these groups G. Furthermore[30], it is known that for any simple gauge group with $\Pi_4(G) = \{0\}$ is free of global gauge anomalies when restricted to any SU(2) subgroup.

3 Global Gauge Anomalies for Groups as Direct Product of SU(2) or SP(2N) in D=4 dimensions

In this section, we will study the possible global gauge anomalies for a class of semisimple gauge groups as direct product of SU(2), or SP(2N) more generally. Our present discussion will be only for the case of D=4 dimensions. We will present our results in terms of propositions, then the proofs and remarks will follow.

Proposition 5. The gauge group $SU(2) \otimes SU(2) \otimes SU(2)$ in the irrep $\omega = (\square, \square, \square)$ or (2,2,2) in terms of dimensions can have Z_2 global gauge anomaly in D=4 dimensions.

Proof. We have H=$SU(2) \otimes SU(2) \otimes SU(2)$ with the relevant homotopy group $\Pi_4(H) = Z_2 \oplus Z_2 \oplus Z_2$. To use our method, we need to find a group G satisfying the embedding condition, namely, $H \subset G$ with $\Pi_4(G) = \{0\}$ being trivial, and G has a irrep $\tilde{\omega}$ such that $\tilde{\omega}$ reduces to ω plus H singlets. It is known that groups SU(N) ($N \geq 8$) contains the H=$SU(2) \otimes SU(2) \otimes SU(2)$ as a subgroup, and $\Pi_4(SU(N)) = \{0\}$ for those $N \geq 3$. The branching rule[31] shows that the fundamental rep $\tilde{\omega} = (\square)$ of SU(8) reduces to the $\omega = (2,2,2)$ upon the reduction of SU(8) to the $SU(2) \otimes SU(2) \otimes SU(2)$. Therefore, we can choose G=SU(8). Actually, we can also use any G=SU(N) ($N \geq 8$). This is due to the fact that[16] for any rep ω of SU(N), there is a rep ω' of SU(N') ($N' > N$) which reduces to the ω plus SU(N) singlets. This implies that we can use any G=SU(N) ($N \geq 8$) in its fundamental rep. However, for simplicity, we will use G=SU(8). Then the possible $H = SU(2) \otimes SU(2) \otimes SU(2)$ global gauge anomalies will appear as the Wess-Zumino term[14−16] corresponding to the embedding of H into G=SU(8) in the fundamental representation.

The corresponding exact homotopy sequence for the fibration $H \to G \to G/H$ is given by[14−20]

$$... \to \Pi_5(G) \to \Pi_5(G/H) \to \Pi_4(H) \to \Pi_4(G) = 0. \tag{8}$$

With $H = SU(2) \otimes SU(2) \otimes SU(2)$ and G=SU(8), this is written as[25]

$$... \to Z \to Z \oplus Z_2 \oplus Z_2 \to Z_2 \oplus Z_2 \oplus Z_2 \to 0. \tag{9}$$

We can calculate the formula for the basic global anomaly coefficient is this case, and it is given by

$$A = exp\{i\pi b Q_3(\Box)\} \tag{10}$$

with b being an integer, or this can be rewritten as

$$A = exp\{i\pi b Q_2(\Box)\} \tag{11}$$

with the even odd rule[30,15−20] $Q_2(\tilde{\omega}) = Q_3(\tilde{\omega})$ (modulo 2) for any $\tilde{\omega}$ of SU(N). The $Q_2(\Box)$ and $Q_3(\Box)$ are the second and third-order Dynkin indices for the fundamental representation $\Box$ of SU(8) respectively with the possible constraint that the SU(8) gauge theory should be free of local anomaly when restricting to the three SU(2) factors on the spacetime S^4 as the boundary of a five-dimensional disc D^5. However, since the three SU(2) factors are automatically free of local gauge anomaly in four dimensions, there is no constraint on the Dynkin index. The integer b is odd or even depending on whether the odd topological number for the Z in $\Pi_5(G/H)$ (up to finite cyclic part) can be mapped to some non-trivial element in $\Pi_4(H)$ or not. It is known that the three Z_2's in the $\Pi_4(H)$ are completely symmetric, the subgroups split in a canonical way, there is a special element $h_1 \oplus h_2 \oplus h_3$ with the h_i (i=1,2,3) being the generators for the three Z_2's respectively. The odd elements of Z in the $\Pi_5(G/H)$ are then indeed mapped to the $h_1 \oplus h_2 \oplus h_3$. We note that in order to obtain the odd topological number for Z in the in $\Pi_5(G/H)$, the corresponding gauge transformation on the D^5 needs to be topologically non-trivial in all the three SU(2) factors when restricting to its boundary S^4. Only in this case, none of the SU(2) factors may be topologically reduced or equivalently to give a factor 2 from the dimension as a multiplicity for the other SU(2) factors, so that the eight-dimensional (irreducible) embedding is topologically effective. This is also because the relevant $\Pi_4(H)$ has one more Z_2 than the torsion of $\Pi_5(G/H)$. A more rigorous proof needs Steenrod algebra and Postnikov systems[25] and is too involved to be given here. We are only interested in the result here. Thus, with $Q_2(\Box) = 1$, the basic global anomaly coefficient in this case is then $A = -1$. Therefore, the $\omega = (2, 2, 2)$ of $SU(2) \otimes SU(2) \otimes SU(2)$ can have Z_2 global gauge anomaly in D=4 dimensions.

Remark. The fact that the h_1, h_2 and h_3 above do not generate global anomalies does not necessarily imply that the $h_1 \oplus h_2 \oplus h_3$ cannot generate a global gauge anomaly, since the relevant homomorphism induced by the exact homotopy sequence is from the $\Pi_5(G/H)$ to the $\Pi_4(H)$ but not reversely.

Proposition 6. The gauge group $SU(2) \otimes SU(2)$ in the irrep $\omega = (\Box, \Box)$ or (2,2) in terms of dimensions can have Z_2 global gauge anomaly in D=4 dimensions.

The proof is essentially the same as that for Propostion 5, except that now H=$SU(2) \otimes SU(2)$, G=SU(N) ($N \geq 4$) in its fundamental rep. For simplicity, we can choose G=SU(4) in the fundamental rep $\tilde{\omega} = (\Box)$ which reduces to the (2,2) rep of H upon the reduction of $G \downarrow H$. The relevant exact homotopy sequence is

$$... \to Z \to Z \oplus Z_2 \to Z_2 \oplus Z_2 \to 0. \tag{12}$$

The $\Pi_4(H)$ again has one more Z_2's than that in the $\Pi_5(G/H)$. The global gauge anomaly coefficient is this case is $A = exp\{i\pi Q_2(\Box)\} = -1$ with $Q_2(\Box) = 1$ for SU(N) $(N \geq 4)$.

Remark. An immediate implication of this proposition is that a left-right symmetric model with an odd number of (2,2) Weyl fermions in $SU(2) \otimes SU(2)$ is inconsistent due to a global gauge anomaly.

Proposition 7. The gauge group $SU(2) \otimes SU(2), ..., \otimes SU(2)$ as N SU(2) factors with $N \geq 4$ in the irrep $\omega = (\Box, \Box, ..., \Box)$ or (2,2,...2) in terms of dimensions have no global gauge anomaly in D=4 dimensions.

Proof: This can be seen in two different cases depending on N=even or odd. If N=2k=even (≥ 4), then we can regard the H as k $(k \geq 2)$ factors of $SU(2) \otimes SU(2)$ in ((2,2),(2,2),..,(2,2)) rep. Since each $SU(2) \otimes SU(2)$ factor can be embedded into SU(4) in its fundamental rep $\Box$ of dimension 4 which reduces to (2,2) upon reduction $SU(4) \downarrow SU(2) \otimes SU(2)$. We can choose G=$SU(4) \otimes SU(4) ... \otimes SU(4)$ as k factors of SU(4) in the irrep (4,4,....,4). In this case, the homotopy group $\Pi_5(G/H) = \sum_1^k \oplus Z \oplus Z_2$. The embedding of H into G leads to k independent Wess-Zumino terms corresponding to the fact that each $SU(2) \otimes SU(2)$ is embedded into a corresponding SU(4). Therefore, the corresponding exact homotopy sequence is decomposed into the direct sum of k independent ones. Each of them has the same structure as that in the Proposition 6. However, the Lie algebra representation for each of the SU(4) factors now is obviously no longer irreducible, since an SU(4) Lie algebra is now from the restriction of that for the k SU(4) factors to a single one. The G as k SU(4) factors is in the rep (4,4,...,4), the Lie algebra elements, say for the first SU(4) factor are of the form $\{L_a \otimes (1)^2_{4,4} \otimes ... \otimes (1)^k_{4,4}\}$ with $(1)^i_{4,4}$ being the 4×4 unit matrix corresponding to the ith SU(4) subalgebra, where the $\{L_a\}$ are the Lie algebra of SU(4) in the fundamental irrep. Let us recall that[14–16] the Wess-Zumino term is enclosed by a trace operation for the matrix rep of the Lie algebra of G, this guarantees that the global anomaly coefficient is in the form of $A = exp(i\pi 2m) = 1$ (m=integers). Therefore, the theory has no global gauge anomalies for $N = 2k \geq 4$. One can also formally see this by regarding the embedding as corresponding to only one Wess-Zumino, then the trace operation in G automatically splits into a summation of k independent terms, each of them corresponds to an independent Wess-Zumino term for the embedding for $SU(2) \otimes SU(2)$ into a SU(4), except that the rep is no longer irreducible as we have seen above. When $N = 2k + 1 \geq 5$, then one can choose G as the semisimple gauge group consisting of k SU(4) factors and one SU(8) factors in the irrep $(\Box, ..., \Box)$ or (4,4,...,4,8) in terms of dimensions. Note that each dimension factor corresponding to the simple ideal is even, for the same reason as above, the global anomaly coefficient must be 1, the theory is free of global gauge anomaly.

Proposition 8. The gauge group $SP(4) \otimes SP(4)$ in the irrep $\omega = (\Box, \Box)$ or (4,4) in terms of dimensions have no global gauge anomaly in D=4 dimensions.

Proof: This can be seen easily by using quite different choices of gauge group G. Let us first see this by using G=SO(10) in a fundamental spinor rep. of dimensions 16. The branching rule[31] shows that a rep 16 of SO(10) reduces to (4,4) upon the reduction of $SO(10) \downarrow SP(4) \otimes SP(4)$, and we also have $\Pi_4(SO(10)) = \{0\}$. Then according to our Proposition 4, the Proposition 8 is immediate. We can also see this by using $G = SU(4) \otimes SU(4)$ in the irrep (4,4) with $\Pi_4(SU(4) \otimes SU(4)) = \{0\}$. The branching rule[16,31] shows that it reduces to the (4,4) for $SP(4) \otimes SP(4)$ upon the reduction of G to it. Then the same proof as that in Proposition 7 applies.

Proposition 9. The gauge group $SP(2N_1) \otimes SP(2N_2) \otimes ... \otimes SP(2N_k)$ with $N_i \geq 2,\ k \geq 2$ in the irrep $\omega = (\Box, \Box, ..., \Box)$ or $(2N_1, 2N_2, ..., 2N_k)$ in terms of dimensions have no global gauge anomaly

in D=4 dimensions.

Proof. We first note the fact that $SP(2N) \subset SU(2N)$, and the branching rule[31] that (2N) reduces to (2N) upon the reduction $SU(2N) \downarrow SP(2N)$. Then the proposition can be easily shown by using G=$SU(2N_1) \otimes SU(2N_2) \otimes ... \otimes SU(2N_k)$ in the irrep $\omega = (\Box, \Box, ..., \Box)$ or $(2N_1, 2N_2, ..., 2N_k)$ in terms of dimensions with $\Pi_4(G) = \{0\}$. Same as in the proof of the Proposition 7, the theory is free of global gauge anomaly.

We have shown several propositions for the possibilities of global gauge anomalies of a semisimple gauge group with more than one SP(2N) ideals. As we will see that some of the above results will be useful to the study of unification gauge groups.

4 Gauge Anomaly for SO(10) and Other Unification Groups and the Selection Rule for Generation Numbers

The global (non-perturbative) gauge anomalies relevant here for SO(10) unification group are more subtle than the usual ones similar to that first noted by Witten[13] with the non-trivial homotopy group $\Pi_4(G)$ for the gauge group G in four dimensions. Since $\Pi_4(SO(10)) = 0$ is trivial, the usual expectation is that there should not be global gauge anomalies. Our idea is to consider the subgroups of SO(10) with non-trivial forth homotopy group. Classically, such non-trivial topological structures are unwrapped in SO(10). In quantum theory, however, if the non-trivial topological structure can generate gauge anomalies, the corresponding large gauge transformations in the subgroup are anomalous or ill-defined. Obviously, the unwrapping then can not be physically well-defined since an ill-defined symmetry transformation in quantum theory cannot be homotopically equivalent to the identity transformation which is always well-defined.

The new global gauge anomalies we noted[1] arise from the restriction of a gauge group G with relevant trivial homotopy group to its gauge subgroups (containing more than one simple ideals with non-trivial forth homotopy group). The example we will discuss is the subgroups of SO(10) due to its crucial importance and relevance to the unification theories. We will now describe our result for the SO(10) gauge theories, and may use the Lie algebras for the discussion of representations. The same notations may be used for the Lie groups and corresponding Lie algebras, no confusion should be caused in our discussion here. We will first show the following proposition.

Proposition 10. The SO(10) gauge group with Weyl fermions in a sixteen-dimensional fundamental spinor (f.s) rep can have a $Z_2 \oplus Z_2$ global gauge anomaly when restricted to the $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$ subgroup obtained through the reduction of its subgroup $SU(2) \otimes SO(7)$ with SO(7) to the subgroup $SU(2) \otimes SU(2) \otimes SU(2)$.

The SO(10) group contains a maximal subgroup $SU(2) \otimes SO(7)$ (The difference between SU(2) and SO(3) in our consideration is immaterial since they have the same forth homotopy group). Restricting the SO(7) to its subgroup $SU(2) \otimes SU(2) \otimes SU(2)$, then we obtain a subgroup $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$. For this subgroup, we have the relevant homotopy group

$$\Pi_4(SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)) = Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \tag{13}$$

The homotopy group topologically classifies the continuous gauge transformations restricted to this subgroup in the compactified spacetime manifold. The non-trivial topological $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ structures exist when the SO(10) gauge theory is restricted to the subgroup. Such a non-trivial topological structure can be unwrapped classically. However, as we have stressed that if the gauge transformations in the subgroup can be anomalous, then in the quantum theory, the gauge transformations in the subgroup are ill-defined, and such an unwrapping cannot be well-defined.

We will show that this is indeed the case, and therefore, the theory has a non-perturbative gauge anomaly. The meaning of the global (non-perturbative) gauge anomaly here may be regarded as that the corresponding gauge transformations cannot be continuously deformed into identity transformations in quantum theory.

The branching rule[31] for a fundamental spinor representation (f.s) or (16) of SO(10) in the above reduction $SO(10) \downarrow SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$ can be written as

$$(16) \rightarrow (2-1-2-2) \oplus (2-2-1-2) \tag{14}$$

in terms of dimensions. To determine the possible global anomalies, we will first consider an irreducible representation (2-1-2-2) in the above branching, then the overall possibilities can be clarified. Obviously, for the irreducible representation (2-1-2-2), it is equivalent to consider the possible global anomaly for the group as three SU(2) factors in the irreducible representation (2-2-2), since the Weyl fermions are invariant under the gauge transformations restricted in the second SU(2) gauge group. Now according to our Proposition 5, such a irrep has a Z_2 global gauge anomaly. Thus, the first irrep (2,1,2,2) in the branching rule can contribute a Z_2 anomaly if it is not canceled by the second irrep (2,2,1,2). For the same reason, the (2,2,1,2) can also have a Z_2 anomaly. Whether the two Z_2 anomalies cancel or not depends on if they have to arise from topologically equivalent gauge transformations in the gauge subgroup $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$. It is obvious that the two large gauge transformations in the subgroup are topologically inequivalent if the first one is topologically non-trivial simultaneously in the first, third, and forth SU(2) ideals in our notation, but the second one is topologically non-trivial simultaneously in the first, second, and forth SU(2) ideals. Namely the two gauge transformations of topological numbers (1,0,1,1) and (1,1,0,1) corresponding to the homotopy group $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ generate two independent Z_2 anomalies. Therefore, the SO(10) theory with Weyl fermions in a fundamental spinor rep when restricted to the $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$ obtained through the reduction as in the above proposition can have $Z_2 \oplus Z_2$ global gauge anomalies. The Proposition 10 is then proved.

The consequence is that[13] the generating functional and the operators invariant under such gauge subgroup cannot be well-defined relative to the relevant large and continuous gauge transformations in the subgroup. The $Z_2 \oplus Z_2$ anomaly may be understood as that when the gauge transformation is topologically non-trivial in three of the SU(2) factors simultaneously but trivial in either the second or the third one in our notation, the fermion measure will change a sign, the quantum theory is then not well-defined[13].

Remark. The two different Z_2 global gauge anomalies for the $Z_2 \oplus Z_2$ arise from the two different irreducible representations in the branching rule eq.(12). They correspond to topologically inequivalent gauge transformations when restricting to the relevant subgroup. As we emphasized, since some 'large' gauge transformations in the subgroup are ill-defined or anomalous, they cannot be unwrapped to the identity in SO(10) in the quantum theory. Therefore, the fact that the SO(10) gauge group with $\Pi_4(SO(10)) = \{0\}$ does not have local gauge anomalies will not contradict to our result. Note that for each of the irreducible representations in eq.(12) (e.g the (2,1,2,2) for the four SU(2) factors), it cannot be embedded into a representation $\tilde{\omega}$ of SO(10) such that the ω reduces to the irreducible representation (2,1,2,2) plus singlets upon the reduction. This is also an explicit example showing that the conventional proposition[16] noted by using the Wess-Zumino term argument does not apply generally to the case in which the relevant subgroup has more than one ideals with non-trivial 2n-th homotopy group in D=2n dimensions and the representation is not irreducible. As we have seen that this is also why a fundamental representation (f.s) of the SO(10) cannot have $SP(4) \otimes SP(4)$ global anomaly, due to the fact that the (f.s) reduces to the sixteen-dimensional *irreducible* representation $(\square, \square)$ upon the reduction $SO(10) \downarrow SP(4) \otimes SP(4)$, the conventional argument of using Wess-Zumino term for the SO(10) may apply. In this case,

the vanishing of SO(10) local anomaly or Wess-Zumino term implies the absence of the relevant $SP(4) \otimes SP(4)$ global anomaly. The problem with embedding a direct sum with more than one irreducible representations of global gauge anomalies is that the anomaly information may not be extracted independently due to the fact that[15−16] the global gauge anomaly for an irreducible representation free of local gauge anomaly can be at most of Z_2 type. Generally, from this point of view for the global gauge anomalies, the restriction of a gauge theory to a gauge subgroup H may not be the same as embedding the gauge subgroup H into the original gauge group G due to the representation condition needed to extract the possible global gauge anomalies.

Remark. We have also noted that the $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$ up to isomorphism is the only possible subgroup of SO(10) in a (f.s) rep having $Z_2 \oplus Z_2$ global gauge anomalies. Another example is the $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$ subgroup obtained through the reduction of the subgroup $SU(2) \otimes SU(2) \otimes SO(6)$ of SO(10) with the SO(6) to two SU(2) factors.

Note also that generally gauge symmetry in a gauge group implies the gauge symmetry in its gauge subgroup (see ref.32 for the other studies related to this property), namely a well-defined gauge theory needs to be well-defined when restricting to its gauge subgroups. In quantum theory, if there are gauge anomalies when restricting to a gauge subgroup, then the gauge theory cannot be well-defined. In conclusion, the SO(10) gauge theories with Weyl fermions in a fundamental spinor representation of dimension 16 have global (non-perturbative) gauge anomalies. The SO(10) has two fundamental spinor representations which are complex conjugate to each other. Our results applies to either one of them.

Denote the numbers of two inequivalent fundamental spinor representations as $N(16)$ and $N(\bar{16})$, obviously we need to have

$$N(16) + N(\bar{16}) = even, \tag{15}$$

in order to cancel out the global gauge anomalies. Consequently, SO(10) unification models with three generations of fermions have global gauge anomalies. We have checked that the adjoint representation of dimensions 45 for the SO(10) will be free of global gauge anomalies for the relevant subgroups. Therefore, our conclusion applies both to the non-supersymmetric SO(10) models and supersymmetric models in which gauginos are in the adjoint representation.

The physics consequences of our result may be of fundamental interest if the SO(10) gauge theories are relevant to the realistic world. Obviously, the SO(10) models and supersymmetric SO(10) unification models need to be modified according to our analysis. In the usual physics convention for the Weyl fermions with the observed three families of leptons and quarks, we have the following selection rule.

Selection Rule For Generation Numbers

In SO(10) and supersymmetric SO(10) unification models, the Weyl fermions need to obey the selection rule written as

$$N_f + N_{mf} = even \geq 4, \tag{16}$$

with the $N_f = N(16)$ and $N_{mf} = N(\bar{16})$ denoting the number of fermion families and the number of mirror fermion families respectively.

Therefore, we predict that there will be at least one more fermion family or at least one mirror fermion family if an SO(10) unification gauge theory is realistic. Where in the content of SO(10) unification, the fourth generation (or a generation of mirror fermions) also includes a right-handed neutrino (or a left-handed mirror neutrino). Mirror fermions have the same $SU(3) \otimes SU_L(2) \otimes U(1)_Y$ quantum numbers as the ordinary fermions except that they have opposite handedness. Usually[33], mirror fermions are considered with three generations. Conventionally, one family of

mirror fermions seems not so motivated. However, our result of the global gauge anomalies shows that it is one of the simple ways to cancel the global anomalies. As in the usual discussions, if there exists fourth generation of fermions with V-A weak interaction, then of course, it seems natural to have either no mirror fermions or four families of mirror fermions. The next possibility is either to have three generations of ordinary fermions and three generations of mirror fermions correspondingly as in the usual discussions of mirror fermions, or to have six generations of fermions with three more repetitions of an ordinary fermion family. If there are mirror fermions, one of the most fundamental consequences will then be that the Lorentz structure of the weak interaction will no longer be chiral with only V-A currents coupling to the W gauge bosons, there will be also V+A piece which though may be very small relevant to the current experimental observation[33]. There has been analysis[33] about the charged and neutral current data suggesting that the possible V+A impurity in the weak amplitudes is typically less that about 10%.

We will now give a brief sketch of some other related physics issues, for details see the relevant references. In the content of the electroweak theory, either an additional generation of fermions or a generation (three generations) of mirror fermions obtain their masses through the electroweak symmetry breaking at the order of about O(300Gev), this will give effects on low energy physics and also subject to both theoretical and experimental constraints. The LEP date set a lower bound for their masses denoted by M_F at about $M_F \geq m_z/2$, namely about half of the Z boson mass. The partial wave unitarity[34] at high energies shows that the masses above about $O(600Gev/\sqrt{N_{DQ}})$ and $O(1Tev/\sqrt{N_{DL}})$ for quarks (or mirror quarks) and leptons (mirror leptons) will signal the breakdown of the perturbation theory, where N_{DQ} denotes the total number of nearly degenerate weak-isospin doublets for quarks and mirror quarks, and similarly with N_{DL} for leptons and mirror leptons. There may be stringent constraint on the masses, mass splitting in a weak-isospin doublet for possible new fermions due to the bound on the correction $\delta\rho$ for the parameter[35,36] $\rho = {m_w}^2/{m_z}^2 \cos^2\theta_w$ from its tree level value in the minimal standard model, as well as for the other precision electroweak parameters[35,37]. It is known that the radiative corrections[35] in perturbation theory can play an active role in this. There are also recent discussions that[36] the possible bound states formed by the exchange of Higgs bosons in the presence of additional heavy fermions may give non-perturbative contribution $\delta\rho < 0$ and cancel the perturbative correction within the current experimental error in the nearly degenerate case, and therefore can relax the constraints for the masses and mass splitting due to the ρ parameter.

An additional family of fermions or mirror fermions may also be constrained by that Yukawa couplings should remain small during the evolution in the perturbative region (about $\alpha_{Yuk} = {\lambda_{Yuk}}^2/4\pi \leq 1$), otherwise its running may induce Landau poles in the one-loop approximation. The presence of these singularities at some scale signals the breakdown of perturbation theory and the probable triviality of the continuum limit. Related to the running of Yukawa couplings and infrared fixed-point solution[38] to the renormalization group equations, there has been discussions[38,39] in supersymmetric unification models with Yukawa coupling unification (e.g. $\lambda_\tau = \lambda_b$ at the unification scale) that only small regions in the $m_t - \tan\beta$ ($\tan\beta = v_{up}/v_{down}$) plane may be allowed (e.g. about $1 \leq \tan\beta \leq 1.5$ or $\tan\beta \geq 40 \pm 10$ with $m_t \leq 175Gev$). The value of $\tan\beta$ can typically effect[38−39] the flavor changing neutral currents in processes like $b \to s\gamma$ and $B\bar{B}$ mixing and to the proton decay[40]. It also constraints on the Higgs boson masses[41] which may be relevant to the LEP II. If there are additional fermions or mirror fermions, one may expect that their presence will also have these typical effects, relevant discussions with Yukawa coupling unification at the unification scale then may need to incorporate them. At least, the possibility of one additional generation of mirror fermions from our motivation in terms of global gauge anomalies sounds quite new. It has been argued[42] in the conventional mirror fermion models (with three generations of mirror fermions corresponding to ordinary observed generations of quarks and leptons) that mirror

doublets should always be assumed degenerate in masses (without considering the non-perturbative effects in ref. 36) in order to reproduce the precision LEP data, and the possible Higgs masses may be in rather restricted regions. We note that mirror fermions of at least three generations usually appear in the particle spectrum of many theories other than some superstring models with family unification, such as those with extended supersymmetry ($N \geq 2$) imposed on a gauge theory, in the Kaluza-Klein theories[43], and some composite models[44]. However, according to our analysis of global anomalies in SO(10) unification gauge theories, one of the interesting models is to have only one generation of mirror fermions besides the three generations of ordinary fermions. In general, one may expect that[33,42] mirror fermions need to mix with the ordinary fermions in order to avoid stable mirror fermions although the mixing may be small. If there exists only one generation of mirror fermions, fundamentally, it is unnatural to assume that it corresponds to a particular family of ordinary fermions. Therefore, this generation of mirror fermions will mix with all the three generations of ordinary fermions, and this then will induce the flavor mixing between the three generations of ordinary fermions also, this seems to provide another origin for the possible flavor mixing and possible CP violations. Moreover, as it is known that the lifetime of heavy neutrinos may subject to cosmological constraint[45] (a suggestion is about $\tau < 10^3 yr(1kev/m_v)^2$) since it may be strongly believed that the age of the universe is greater than 10^{10} years. Our selection rule may have consequences on the structure of the universe. Our rule for the generation numbers can be of fundamental interest and importance. We will conclude this section with the following related propositions and discussions.

Proposition 11. The SO(10) gauge group with Weyl fermions in a fundamental spinor (f.s) rep can have a $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ global (non-perturbative) gauge anomaly when restricting to $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2) \otimes$ obtained through the reduction of the subgroup $SP(4) \otimes SP(4)$ of SO(10) with each SP(4) to two SU(2) factors.

Proof: The reduction of SO(10) to its subgroup $SP(4) \otimes SP(4)$ has the branching rule[31] $(f.s) \rightarrow (4, 4)$. Upon the reduction of SP(2) to $SU(2) \otimes SU(2)$, we have $(4) \rightarrow (1, 2) \oplus (2, 1)$. Thus, the branching rule for the reduction $SO(10) \downarrow SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2) \otimes$ in this case is given by

$$(16) \rightarrow (1, 2, 1, 2) \oplus (1, 2, 2, 1) \oplus (2, 1, 1, 2) \oplus (2, 1, 2, 1). \tag{17}$$

According to the proposition 6, each of the irrep in the branching rule above may contribute a Z_2 anomaly. The theory obviously has $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ global gauge anomalies corresponding the gauge transformations with the topological numbers (0,1,0,1), (0,1,1,0), (1,0,0,1), (1,0,1,0) respectively. The proposition is then proved.

Remark. Obviously, the $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ global gauge anomalies for the SO(10) in a fundamental rep cancel out if the total number of families for fermions and mirror fermions is even. Therefore, although the SO(10) can have different types of global gauge anomalies, but they all cancel out if our selection rule for the generation numbers is satisfied.

Proposition 12. The vector representation of dimension 10 for the SO(10) can have global gauge anomalies when restricting to certain semisimple subgroups as product of SU(2) factors through many reductions. But they all cancel out if the total number of the vector representations are even. Some examples are given below.
(1)$Z_2 \oplus Z_2$ global anomalies when restricting to the $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$ through the reduction of subgroup $SP(4) \otimes SP(4)$ of SO(10) with each SP(4) to two SU(2) factors;
(2) $Z_2 \oplus Z_2$ global gauge anomalies when restricting to a $SU(2) \otimes SU(2)$ gauge subgroup through the following reductions:
(2a)The $SU(2) \otimes SU(2)$ from the reduction of either one of the SP(4) factors in (1);

(2b) the $SU(2)\otimes SU(2)\otimes SO(6)$ ($SO(6)\cong SU(4)$) subgroup of SO(10) to the two SU(2) factors; (2c) by the reduction of the subgroup $SU(2)\otimes SO(7)$ of SO(10) to $SU(2)\otimes SU(2)\otimes SU(2)\otimes SU(2)$, the relevant two SU(2) factors both are in the fundamental representation in one of the irreducible representations (2-2-1) in the branching rule for the reduction of the SO(7) to $SU(2)\otimes SU(2)\otimes SU(2)$. Moreover, the SO(10) group in low-dimensional representations of dimensions 45, 54, 120 etc. will not have global gauge anomalies.

By using the branching rules[31] for these reductions, and our propositions in section 3, it is straightforward to verify the above proposition. We only show this for the case (1) as an example. Upon the reduction $SO(10)\downarrow SP(4)\otimes SP(4)$, we have $(10)\rightarrow(5,1)\oplus(1,5)$. Since $(5)\rightarrow(2,2)\oplus(1,1)$ upon $SP(4)\downarrow SU(2)\otimes SU(2)$. This gives $(10)\rightarrow(2,2,1,1)\oplus(1,1,2,2)\oplus 2(1,1,1,1)$ upon the reduction $SO(10)\downarrow SU(2)\otimes SU(2)\otimes SU(2)\otimes SU(2)$. According to the proposition 6, the theory restricted to the $SU(2)\otimes SU(2)\otimes SU(2)\otimes SU(2)$ obviously can have $Z_2\oplus Z_2$ gauge anomalies.

We especially emphasize that since in the supersymmetric SO(10) models, the gauginos are in the 45-dimensional adjoint representation which is free of gauge anomalies, our selection rule applies both to non-supersymmetric SO(10) and supersymmetric SO(10) unification theories. Before the summary of our conclusions, we will also briefly discuss about some other gauge groups relevant to unification theory. It is straight to verify the results with the relevant branching rules and our propositions in the section 3.

Remark. Obviously, the global gauge anomalies in a fundamental spinor representation cannot be canceled by adding more vector representations and vice versa. It is a typical feature that the branching rules become much more involved when the dimension goes higher, if there were other possible anomalies they would be very dependent on the reduction procedure, global anomalies from representations of different dimensions may not cancel each other. At least, we checked up to dimensions of several hundred, no other higher irreducible representations can have the same possibilities for the global anomalies as that of a fundamental spinor representation. Our selection rule for the generation numbers in is realistically general.

Remark. For superstring theory with gauge group $E_8\times E_8'$, a compactification of the heterotic string is $E_6\times E_8'$, N=1 supersymmetric Yang-Mills theory in four dimensions[42]. In the E_6 sector, the left-handed Weyl fermions are in the real representation $\{78\}\oplus 3\times\{27\}\oplus 3\times\{\bar{27}\})\oplus 8\times\{1\}$ of the E_6 in terms of the dimensions. In four dimensions, the E_6 is a local anomaly-free group with $\Pi_4(E_6)=0$ being trivial. We can show that this representation will not have a global anomaly when restricting to a subgroup with non-trivial forth homotopy group. In this case of E_6 upon the reduction to SO(10), it can also be seen more obviously by our analysis. Upon the reduction $E_6\downarrow SO(10)$, $78\rightarrow 45\oplus 16\oplus\bar{16}\oplus 1$, and $27\rightarrow 16\oplus 10\oplus 1$ and correspondingly for the $\bar{27}$. After the decomposition, there is a $\bar{16}$ for each 16 and the 10 also appear in pairs. Therefore, there will be no global gauge anomalies upon reduction to SO(10). One can also see explicitly that the theory have no global gauge anomaly for the other possible gauge subgroups with non-trivial forth homotopy group. Therefore, the relevant heterotic string theory is free of both local and global gauge anomalies. An E_6 unification gauge theory with even total number for 27 and its complex conjugate $\bar{27}$ is free of global gauge anomaly.

Remark. For the SU(5) and supersymmetric SU(5) theories, we have shown that the relevant Weyl fermion representations (e.g. $5\oplus\bar{10}$) free of local gauge anomaly are also free of global gauge anomaly for the gauge subgroups (such as SP(4) and $SU(2)\times SU(2)$ etc.),as well as[30,16] SU(2) with non-trivial forth homotopy group. Moreover, for the relevant representations of E_8 (adjoint or fundamental rep), SU(4) (for example $8\{4\}\oplus\bar{10}$), and SU(6) gauge groups free of local gauge anomaly, with branching rules and the propositions in section 3 it can be verified that they are free of global gauge anomaly for the other subgroups with non-trivial relevant homotopy group. However, we will not present the details further here.

5 Conclusions

We have studied the possibilities of global (non-perturbative) gauge anomalies in a class of gauge groups. In particular, we investigated the possible global gauge anomalies in a class of semisimple groups as products of SU(2) and more generally SP(2N) (N=rank) groups with non-trivial forth homotopy groups. The results are applied to the determination of possible global gauge anomalies in unification groups. Based on the fact that[1] if a gauge theory with Weyl fermions has global gauge anomalies, then the anomalous or ill-defined large gauge transformations cannot be unwrapped in quantum theory when embedded into a larger group G with $\Pi_4(G) = \{0\}$ although the unwrapping is topologically well-defined classically, we discussed extensively about the global gauge anomalies in SO(10) unification theories containing Weyl fermions in a (f.s) rep. when restricted to a subgroup $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$. The physical consequence is our selection rule1 $N_f + N_{mf} = even \geq 4$ for generation numbers in SO(10) and supersymmetric SO(10) unification theories with N_f and N_{mf} denoting the family numbers for the ordinary fermions and mirror fermions. We have also briefly discussed about the other gauge groups relevant to unification theory in this connection. Another result is that an odd number of (2,2) Weyl fermions in $SU(2) \times SU(2)$ (e.g left-right symmetric models) is inconsistent due to a global gauge anomaly. We expect that our propositions and the discussions may be useful for the other general study of non-abelian gauge theories also.

Finally, we note that the non-trivial forth homotopy group of SP(2N) gauge groups may induce spontaneous T, or CP, violations etc. in such gauge theories as noted by the present author[47–49]. Therefore, our study in connection to Lie groups is significant to the understanding of gauge symmetries as well as spacetime symmetries[2,50].

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